Some fixed point theorems in ordered metric spaces having t-property

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Abstract

In this study, we introduce the new concept of θ_t-contractive and (φ, θ_t)-contractive mappings in ordered metric spaces having t-property. We obtain these theorems without requiring that the metric spaces are complete. Finally, we present some examples to illustrate the new theorems are applicable.

2000 Mathematics Subject Classification. **47H10**. 54H25.
Keywords. fixed point theorem, θ_t-contractive mapping, t-property.

1 Introduction

Existence of fixed points for contraction mappings in complete metric spaces was introduced by Banach [1], also known as the Banach contraction principle, which claims that if (Y, d) is complete metric spaces and $S : Y \to Y$ is a contractive mapping $d(Sz, Sw) \leq Ld(z, w)$ for all $z, w \in Y$ and $L \in [0, 1)$. Several authors introduced various extensions and generalizations of the Banach contraction principle. For example in 2014 Jleli and Samet [2] introduced the following θ-contractive. Defined by Θ is set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing;

(Θ_2) for each sequence $\{k_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(k_n) = 1$ if and only if $\lim_{n\to\infty} k_n = 0^+$;

(Θ_3) there exist $c \in (0, 1)$ and $d \in (0, \infty]$ such that $\lim_{k\to 0^+} \frac{\theta(k)-1}{k^c} = d$.

According to [3], define by φ the set of functions $\varphi : [1, \infty) \to [1, \infty)$ satisfying the following conditions:

(φ_1) $\varphi : [1, \infty) \to [1, \infty)$ is non-decreasing;

(φ_2) for each $k > 1$, $\lim_{n\to\infty} \varphi^n(k) = 1$;

(φ_3) φ is continuous $[1, \infty)$.

Lemma 1.1. [3] If $\varphi \in \varphi$, then $\varphi(1) = 1$, and for each $k > 1$, $\varphi(k) < k$.

Ran and Reurings [4] introduced a fixed point result on a partially ordered metric space. Thereafter, some results, various extensions and generalizations on partially ordered can be found in [5, 6, 7, 8, 9].

Rashid, et al. [10], the completeness of the metric space is removed in the given results. To overcome this lack, they introduced that space has the t-property.

Tbilisi Mathematical Journal Special Issue (8 - 2021), pp. 219–226.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 18 September 2020.
Accepted for publication: 23 December 2020.

Definition 1.2. [10] Let $(Y, \preceq)$ be an ordered set and $z, w \in Y$. z is said to be strict upper bound of w, if $w \preceq z$ and $z \neq w$. We donete it by $w \prec z$.

Definition 1.3. [10] Let $(Y, \preceq, d)$ be an ordered metric space. Y has the t-property if every strictly increasing Cauchy sequence $\{z_n\}$ in Y has a strict upper bound in Y, i.e., there exists $e \in Y$ such that $z_n \prec e$.

In this article, following by Rashid, et al. [10], Zheng et al.[3], Jleli and Samet [2], we introduce some fixed point theorems for new contractive mappings in partially ordered metric spaces having t property. We obtain these theorems without requiring that the metric spaces are complete.

2 Main results

In this section, we present our main results. First, we give the following θ_t-contractive mapping.

Definition 2.1. Let $(Y, \preceq, d)$ be an ordered metric space and $S : Y \to Y$ be a mapping and $\theta \in \Theta$. Then we say that S is θ_t-contractive mapping if there exists $\delta \in (0, 1)$ such that for all $z, w \in Y$ with $z \neq Sz$, $w \neq Sw$ and $z \prec w$, we have

$$\theta(d(w, S(w))) \leq [\theta(d(z, S(z)))]^{\delta}. \tag{2.1}$$

Theorem 2.2. Let $(Y, \preceq, d)$ be an ordered metric space having t property and $S : Y \to Y$ be a θ_t-contractive mapping. Assume that S is non-decreasing and there exists $z_0 \in Y$ such that $z_0 \preceq S(z_0)$. Then S has at least one fixed point.

Proof. We have $z_0 \in Y$ such that $z_0 \preceq S(z_0)$. If $z_0 = S(z_0)$ then the proof is completed. Now, choose $z_1 = S(z_0)$ such that $z_0 \prec z_1$. Since S is monotonicity, we have $S(z_0) \preceq S(z_1)$, that is, $z_1 \preceq S(z_1)$. If $z_1 = S(z_1)$ then the proof is complete. Similarly, choose $z_2 = S(z_1)$ such that $z_1 \prec z_2$. Since S is monotonicity, we have $S(z_1) \preceq S(z_2)$, that is, $z_1 \preceq S(z_1)$. Continuous this condition, we have a strictly increasing sequence $\{z_n\}$ in Y such that $z_{n+1} = S(z_n)$. From $z_0 \prec z_1$ and using (2.1), we obtain

$$\theta(d(z_1, S(z_1))) \leq [\theta(d(z_0, S(z_0)))]^{\delta}. \tag{2.2}$$

Similarly, from $z_1 \prec z_2$ and using (2.1), we obtain

$$\theta(d(z_2, S(z_2))) \leq [\theta(d(z_1, S(z_1)))]^{\delta} \leq [\theta(d(z_0, S(z_0)))]^{\delta^2}. \tag{2.3}$$

From the above inequalities, we have

$$\begin{aligned} \theta(d(z_n, S(z_n))) &\leq [\theta(d(z_{n-1}, S(z_{n-1}))]^{\delta} \\ &\leq [\theta(d(z_{n-2}, S(z_{n-2}))]^{\delta^2} \\ &\ \vdots \\ &\leq [\theta(d(z_0, S(z_0))]^{\delta^n}. \end{aligned} \tag{2.4}$$

On taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \theta(d(z_n, S(z_n))) = 1, \tag{2.5}$$

which implies from (Θ_2) that

$$\lim_{n\to\infty} d(z_n, S(z_n) = 0^+.$$

From condition (Θ_3), there exists $p \in (0,1)$ and $Q \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(z_n, S(z_n)) - 1}{(d(z_n, S(z_n)))^p} = Q. \tag{2.6}$$

Suppose that $Q < \infty$. Then, let $R = \frac{Q}{2} > 0$. We get, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d(z_n, S(z_n))) - 1}{(d(z_n, S(z_n)))^p} - Q \right| \leq R, \quad \text{for all} \quad n \geq n_0.$$

Which implies that

$$\frac{\theta(d(z_n, S(z_n))) - 1}{(d(z_n, S(z_n)))^p} \geq Q - R = R, \quad \text{for all} \quad n \geq n_0.$$

Subsequently, for all $n \geq n_0$, we obtain

$$n(d(z_n, S(z_n)))^p \leq Tn[\theta(d(z_n, S(z_n))) - 1],$$

where $T = \frac{1}{R}$. Suppose that $Q = \infty$. Let $R > 0$ be an arbitrary positive number. We get, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(d(z_n, S(z_n))) - 1}{(d(z_n, S(z_n)))^p} \geq R,$$

for all $n \geq n_0$. Which implies that for all $n \geq n_0$,

$$n(d(z_n, S(z_n)))^p \leq Tn[\theta(d(z_n, S(z_n))) - 1],$$

where $H = \frac{1}{R}$. Thus, in two cases, there exists $T > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$n(d(z_n, S(z_n)))^p \leq Tn[\theta(d(z_n, S(z_n))) - 1].$$

Using (2.4), we have

$$n(d(z_n, S(z_n)))^p \leq Tn([\theta(d(z_0, S(z_0)))]^{\delta^n} - 1), \tag{2.7}$$

for all $n \geq n_0$. Letting $n \to \infty$ in (2.7), we get

$$\lim_{n\to\infty} n(d(z_n, S(z_n)))^p = 0.$$

Therefore, there exists $n_1 \in \mathbb{N}$ such that

$$d(z_n, S(z_n)) \leq \frac{1}{n^{\frac{1}{p}}}, \quad \text{for all} \quad n \geq n_1. \tag{2.8}$$

For all $n, m \in \mathbb{N}$ with $m > n \geq n_1$. We have

$$\begin{aligned} d(z_n, z_m) \leq& d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + \cdots + d(z_{m-1}, z_m) \\ =& d(z_n, S(z_n)) + d(z_{n+1}, S(z_{n+1})) + \cdots + d(z_{m-1}, S(z_{m-1})) \\ =& \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{p}}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{p}}} \to 0. \end{aligned}$$

This yields that $\{z_n\}$ is a strictly increasing Cauchy sequence in Y which has t-property. Hence, there exists $e \in Y$ such that $z_n \prec e$. If $S(e) = e$, then, the proof is complete. Suppose on contrary that

$$\begin{aligned} \theta(d(e, S(e))) \leq& [\theta(d(z_n, S(z_n)))]^{\delta} \\ \leq& [\theta(d(z_{n-1}, S(z_{n-1})))]^{\delta^2} \\ &\vdots \\ \leq& [\theta(d(z_0, S(z_0)))]^{\delta^{n+1}}. \end{aligned}$$

On taking limit as $n \to \infty$, we obtain $d(e, S(e)) = 0$. Therefore, we get $e = S(e)$. Moreover let f be any strict upper bound of $e \in Y$, then $e \prec f$. Using (2.1), we obtain

$$\begin{aligned} \theta(d(f, S(f))) \leq& [\theta(d(e, S(e)))]^{\delta} \\ \leq& \theta(d(e, S(e))). \end{aligned}$$

Thus we obtain $f = S(f)$, that is, f is also a fixed point of S and so the proof is complete.

Q.E.D.

Definition 2.3. Let $(Y, \preceq, d)$ be an ordered metric space and $S : Y \to Y$ be a mapping and $\theta \in \Theta$. Then we say that S is (φ, θ_t)-contractive mapping if there exists $\varphi \in \varphi$ such that for all $z, w \in Y$ with $z \neq Sz$, $w \neq Sw$ and $z \prec w$, we have

$$\theta(d(w, S(w))) \leq \varphi[\theta(d(z, S(z)))]. \tag{2.9}$$

Theorem 2.4. Let $(Y, \preceq, d)$ be an ordered metric space having t property and $S : Y \to Y$ be a (φ, θ_t)-contractive mapping. Assume that S is non-decreasing and there exists $z_0 \in Y$ such that $z_0 \preceq S(z_0)$. Then S has at least one fixed point.

Proof. We have $z_0 \in Y$ such that $z_0 \preceq S(z_0)$. If $z_0 = S(z_0)$ then, the proof is complete. Now, choose $z_1 = S(z_0)$ such that $z_0 \prec z_1$. Since S is monotonicity, we have $S(z_0) \preceq S(z_1)$, that is $z_1 \preceq S(z_1)$. If $z_1 = S(z_1)$ then, the proof is complete. Similarly, choose $z_2 = S(z_1)$ such that $z_1 \prec z_2$. Since S is monotonicity, we have $S(z_1) \preceq S(z_2)$, that is, $z_1 \preceq S(z_1)$. Continuous this condition, we have a strictly increasing sequence $\{z_n\}$ in Y such that $z_{n+1} = S(z_n)$. From $z_0 \prec z_1$ and using (2.9), we obtain

$$\theta(d(z_1, S(z_1))) \leq \varphi[\theta(d(z_0, S(z_0)))]. \tag{2.10}$$

Similarly, from $z_1 \prec z_2$ and using (2.9), we obtain

$$\theta(d(z_2, S(z_2))) \leq \varphi[\theta(d(z_1, S(z_1)))]. \tag{2.11}$$

From the above inequalities, we have

$$\begin{aligned}\theta(d(z_n, S(z_n))) \leq& \varphi[\theta(d(z_{n-1}, S(z_{n-1})))]\\ \leq& \varphi^2[\theta(d(z_{n-2}, S(z_{n-2})))]\\ &\vdots\\ \leq& \varphi^n[\theta(d(z_0, S(z_0)))]. \end{aligned} \tag{2.12}$$

On taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \theta(d(z_n, S(z_n))) = 1, \tag{2.13}$$

which implies from (Θ_2) that

$$\lim_{n\to\infty} d(z_n, S(z_n) = 0^+.$$

From condition (Θ_3), there exists $p \in (0,1)$ and $Q \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(z_n, S(z_n)) - 1}{(d(z_n, S(z_n)))^p} = Q. \tag{2.14}$$

Suppose that $Q < \infty$. Then, let $R = \frac{Q}{2} > 0$. We get, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d(z_n, S(z_n))) - 1}{(d(z_n, S(z_n)))^p} - Q \right| \leq R, \quad \text{for all} \quad n \geq n_0.$$

Which implies that

$$\frac{\theta(d(z_n, S(z_n))) - 1}{(d(z_n, S(z_n)))^p} \geq Q - R = R, \quad \text{for all} \quad n \geq n_0.$$

Subsequently, for all $n \geq n_0$, we obtain

$$n(d(z_n, S(z_n)))^p \leq Tn[\theta(d(z_n, S(z_n))) - 1],$$

where $T = \frac{1}{R}$. Suppose that $Q = \infty$. Let $R > 0$ be an arbitrary positive number. We get, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(d(z_n, S(z_n))) - 1}{(d(z_n, S(z_n)))^p} \geq R,$$

for all $n \geq n_0$. Which implies that for all $n \geq n_0$,

$$n(d(z_n, S(z_n)))^p \leq Tn[\theta(d(z_n, S(z_n))) - 1],$$

where $T = \frac{1}{R}$. Thus, in two cases, there exists $T > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$n(d(z_n, S(z_n)))^p \leq Tn[\theta(d(z_n, S(z_n))) - 1].$$

Using (2.12), we have

$$n(d(z_n, S(z_n)))^p \leq Tn(\varphi^n[\theta(d(z_0, S(z_0)))] - 1), \tag{2.15}$$

for all $n \geq n_0$. Letting $n \to \infty$ in (2.15), we get

$$\lim_{n\to\infty} n(d(z_n, S(z_n)))^p = 0.$$

Therefore, there exists $n_1 \in \mathbb{N}$ such that

$$d(z_n, S(z_n)) \leq \frac{1}{n^{\frac{1}{p}}}, \quad \text{for all} \quad n \geq n_1. \tag{2.16}$$

For all $n, m \in \mathbb{N}$ with $m > n \geq n_1$. We have

$$\begin{aligned} d(z_n, z_m) \leq& d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + \cdots + d(z_{m-1}, z_m) \\ =& d(z_n, S(z_n)) + d(z_{n+1}, S(z_{n+1})) + \cdots + d(z_{m-1}, S(z_{m-1})) \\ =& \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{p}}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{p}}} \to 0. \end{aligned}$$

This yields that $\{z_n\}$ is a strictly increasing Cauchy sequence in Y which has t-property. Hence, there exists $e \in Y$ such that $z_n \prec e$. If $S(e) = e$, then, the proof is complete. Assume on contrary that

$$\begin{aligned} \theta(d(e, S(e))) \leq& \varphi[\theta(d(z_n, S(z_n)))] \\ \leq& \varphi^2[\theta(d(z_{n-1}, S(z_{n-1})))] \\ &\vdots \\ \leq& \varphi^{n+1}[\theta(d(z_0, S(z_0)))]. \end{aligned}$$

On taking limit as $n \to \infty$ we obtain $d(e, S(e)) = 0$. Therefore we get $e = S(e)$. Moreover let f be any strict upper bound of $e \in Y$, then $e \prec f$. Using (2.9), we obtain

$$\begin{aligned} \theta(d(f, S(f))) \leq& \varphi[\theta(d(e, S(e)))] \\ <& \theta(d(e, S(e))). \end{aligned}$$

Thus we obtain $f = S(f)$, that is, f is also a fixed point of S and so the proof is complete. Q.E.D.

3 Examples

Example 3.1. Let $Y = \{c_r : c_{r+1} = 5c_r + 1, \quad \text{for} \quad r \geq 0 \quad \text{and} \quad c_0 = -1\} \cup (0, 1] \cap \mathbb{Q}$ and $d(z, w) = |z - w|$. So, $Y = \{\cdots, -94, -19, -4, -1\} \cup (0, 1] \cap \mathbb{Q}$. Define an order relation $\preceq$ on

Y, where $\leq$ is usual order. Obviously, $(Y, \preceq, d)$ is not complete but has the t-property. Define a mapping $S : Y \to Y$ by

$$S(z) = \begin{cases} 5z + 1, & z \leq -1 \\ z, & \text{otherwise.} \end{cases}$$

Then, S is non-decreasing. We claim that S are θ_t-contractive and (φ, θ_t)-contractive mappings with $\theta(p) = e^{pe^p}$, $\delta = e^{-4(w-z)}$ and

$$\varphi(k) = \begin{cases} 1, & k \in [1, 2] \\ k - 1, & k \in [2, \infty). \end{cases}$$

To see this, let $z, w \in Y$ with $z < w$. If $w \geq -1$ then $S(w) = w$, that is, $d(w, S(w)) = 0$ and so the proof is completed. Suppose that $z < w \leq -1$. So, $d(w, S(w)) = -(4w + 1)$ and $d(z, S(z)) = -(4z + 1)$ Thus, Theorem 2.2 and Theorem 2.4 are satisfied. Moreover, we obtain $d(S(z), S(w)) > d(z, w)$. Then, using (Θ_1) we obtain $\theta(d(S(z), S(w))) > [\theta(d(z, w))]^{\delta}$ also, by (φ_1), $\theta(d(S(z), S(w))) > \varphi[\theta(d(z, w))]$. Therefore, S are not θ-contractive and (φ, θ)-contractive mappings.

Example 3.2. Let $Y = \{0, \pm 1, \pm 2, \cdots\}$ and $d(z, w) = |z - w|$. Define an order relation $\preceq$ on Y, where $\leq$ is usual order. Obviously, $(Y, \preceq, d)$ is not complete but has the t-property. Define a mapping $S : Y \to Y$ by

$$S(z) = \begin{cases} 4z, & z < 0 \\ z, & z \geq 0 \end{cases}$$

Then, S is non-decreasing. Let's take the $\varphi(k)$ function as in example 3.1. We claim that S are θ_t-contractive and (φ, θ_t)-contractive mappings with $\theta(p) = e^{pe^p}$, $\delta = e^{-\frac{1}{2}}$. To see this, let $z, w \in Y$ with $z < w$. If $w - z \geq 1$ then $S(w) = w$, that is, $d(w, S(w)) = 0$ and so the proof is completed. Suppose that $z < w < 0$. So, $d(w, S(w)) = -3w$ and $d(z, S(z)) = -3z$ Similarly, Theorem 2.2 and Theorem 2.4 are satisfied. Moreover, since a similar process is done as in example 3.1, S are not θ-contractive and (φ, θ)-contractive mappings.

These examples show the new class θ_t-contractive mapping is not included in θ-contractive mapping.

4 Conclusion

Jleli and Samet [2] introduced a new type of contractions called θ-contraction. Rashid, et al. [10], the completeness of the metric space is removed in the given results. To overcome this lack, they introduced that space has the t-property. In this study, we denote a new approach to θ-contraction mappings by combining the ideas of Rashid, et al., Zheng et al.[3], Jleli and Samet. We establish the concept of θ_t-contractive and (φ, θ_t)-contractive mappings in ordered metric spaces without requiring that the metric space is complete, but using the concept of the t-property. We give some examples to illustrate the new theorems are applicable.

Acknowledgments

The authors wish to thank the referees for their careful reading of the manuscript and valuable suggestions.

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